Attractor dimension and dissipative scales in MHD turbulence

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1 introduction

The aim of the present work is to derive rigorous estimates for turbulent MHD flow quantities such as the size and anisotropy of the dissipative scales, as well as the transition between 2d and 3d state. To this end, we calculate an upper bound for the attractor dimension of the motion equations, which indicates the number of modes present in the fully developed flow. This method has already been used successfully to derive such estimates for 2d and 3d hydrodynamic turbulence , as in [3]. We tackle here the problem of a periodic flow in the 3 spatial directions, to which a permanent magnetic field is applied. In addition, the detailed study of the dissipation operator provides more indications about the structure of the flow.

2 The Navier-Stokes equation as a dynamical system

We shall first explain the interest of studying the dynamical system associated with the Navier-Stokes equations. The quantity we are mostly interested in is the set of functions which "attracts" any initial flow, in the sense of the limit when the time t tends to infinity. Indeed, the dimension of this so-called global attractor is known to be high for turbulent flows, but finite under the assumption that the Navier-Stokes equations do not produce any finite time singularity [3]. Physically, this indicates that an established homogeneous turbulent flow includes a finite number of vortices, which therefore cannot be smaller than the ratio of the the volume of the physical domain by the number of modes, precisely given by the attractor dimension d_{att}. Evaluating an upper bound is thus a way to derive a lower bound for the size of the dissipative scales. This will be our purpose from now on.

To calculate the attractor dimension of a dynamical system (defined by an evolution equation of the kind $\frac{\partial}{\partial t}\mathbf{u} = \mathbf{F}(\mathbf{u})$), we consider a solution $\mathbf{u}$ located on the attractor and an arbitrary number N of small independent disturbances $\delta\mathbf{u}_i / i \in \{1..N\}$. Note that "small" is relative to the norm defined in the phase space, which is a space of functions in the case of the Navier-Stokes system. The subset spanned by these N independent vectors evolves so that it is located within the attractor at infinite time. Therefore, if $N > d_{att}$, the N-dimensional volume of this subset, defined as

$$V_N(t) = |\delta\mathbf{u}_1 \times .. \times \delta\mathbf{u}_N|$$

tends to 0 when t tends to infinity. This latter property is expressed by Constantin and Foias theorem [2].
In the vicinity of the attractor, the evolution operator can be linearised as $\mathbf{F}(\mathbf{u}) = \mathbf{A}\mathbf{u} + O(\|\delta\mathbf{u}\|^2)$ so that the variations of $V(t)$ are exponential:

$$V_N(t) = V_N(0)\exp(t\langle \mathrm{Tr}(\mathbf{A_N})\rangle) \quad (1)$$

The subscript N stands for projection of operator into N-dimensional subsets of the phase space. If $\mathrm{Tr}(A_N)$ is positive for at least one choice of N disturbances, then N is an upper bound for the attractor's dimension because at least one N-volume would expand (see (1)). We shall therefore look for the maximum trace of the 0 Navier-Stokes equations for any arbitrary integer N.
In the case of a uniform and permanent magnetic field $\mathbf{B}$ aligned with the z axis, the latter can be 4:

$$\nabla.\delta\mathbf{u} = 0 \quad (2)$$

$$\frac{\partial}{\partial t}\mathbf{u} = \frac{\nu U}{L^2}\left[\underbrace{Re\,(-\mathbf{u}.\nabla\delta\mathbf{u} - \delta\mathbf{u}.\nabla\mathbf{u})}_{\mathcal{I}(\mathbf{u})\delta\mathbf{u}} + \underbrace{Ha\delta\mathbf{j}\times\mathbf{B} + \Delta\delta\mathbf{u}}_{\mathbf{D}_N\delta\mathbf{u}}\right] \quad (3)$$

where $\mathbf{j}$ denotes the electric current. σ, ρ and ν are the fluid electrical conductivity, density and viscosity. The system is completed with periodical boundary conditions of period L in the 3 directions of space. The square of the Hartmann number $Ha = LB\sqrt{\frac{\sigma}{\rho\nu}}$ expresses the ratio of the Lorentz force to viscosity, and $Re = \frac{UL}{\nu}$ is the Reynolds number.
The total rate of expansion results from the competition between the non linear terms which tend to expand the initial volume in the phase space, and viscous and Joule dissipations which tend to contract it.

Recently, [5] has derived a lower bound for the trace of the operator associated with the inertial terms in any N-dimensional subset, of the form

$$\mathrm{Tr}(\mathcal{I}(\mathbf{u})) \leq N Re^2 \tag{4}$$

in the case of a periodical flow in the 3 spatial dimensions. In order to derive an estimate for the attractor dimension in the MHD case, our main task then consists in finding the minimum of the trace of the dissipation operator on all N-dimensional subspace, for arbitrary values of N.

3 Properties of the modes minimising the dissipation

We shall now look for the set of N modes that achieve the minimum dissipation for any value of N and exhibit a few important properties of these modes. The dissipation operator is compact and self-adjoint, so its trace expresses as the sum of its eigenvalues. The next step is then to solve the eigenvalue problem of the dissipation operator and to sort the eigenvalues in ascending order. The sum of the N first actually achieves the minimum of the trace over all N-dimensional subset of the phase space. The three spatial component of the eigenvector appear to be of the form:

$$U(x,y,z) = \sin(2\pi k_x)\sin(2\pi k_y)\sin(2\pi k_z)/(k_x,k_y,k_z) \in N^3 (k_x,k_y,k_z) \neq (0,0,0) \tag{5}$$

The eigenvalue associated to the mode (k_x, k_y, k_z) expresses its dissipation rate and writes:

$$\lambda(k_x,k_y,k_z) = k_x^2 + k_y^2 + k_z^2 - Ha^2 \frac{k_z^2}{k_x^2+k_y^2+k_z^2} \tag{6}$$

The function $\lambda(k_x, k_y, k_z)$ is convex so that if λ_{max} is the largest eigenvalue (corresponding to the N^{th} mode), all modes associated to smaller eigenvalues are located inside the area delimited by the curve $\lambda(k_\perp, k_z) = \lambda_{max}$ in the $(k_\perp, k_z)$ plane, where $k_\perp = \sqrt{k_x^2 + k_y^2}$, as shown on figure 1 . The knowledge of the iso-λ_{max} curve also provides the maximum values of the modes in the direction of the magnetic field $k_{z_{max}}$ and in the orthogonal direction $k_{\perp_{max}}$, the ratio of which is an indication of the anisotropy of the small scales.

These particularities can be used to calculate the N first modes and the associated trace of the dissipation as a function of N and Ha. This can be done in the general case, using an iterative algorithm implemented on a computer. The result is shown on figure 2.

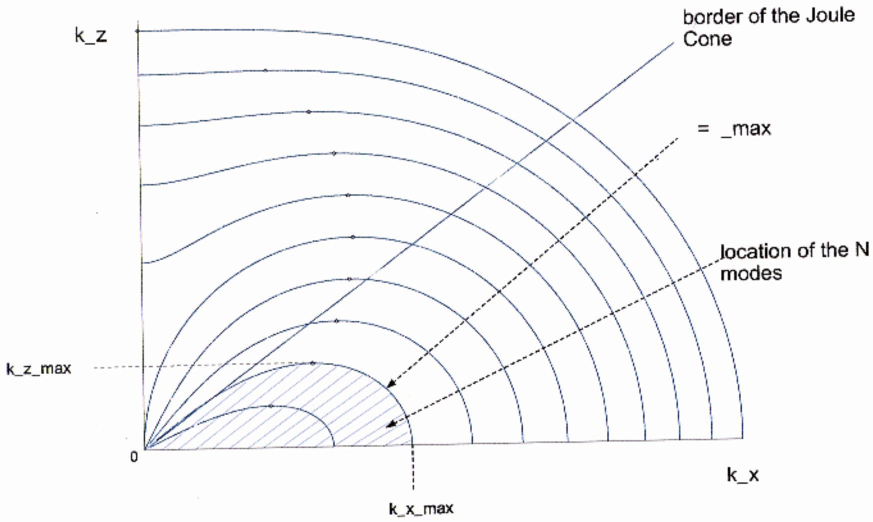

Figure 1: Iso-λ curves. Quarter-circle represent quasi-isotropic flows, whereas, elongated shapes reflect a dominant Lorentz force.

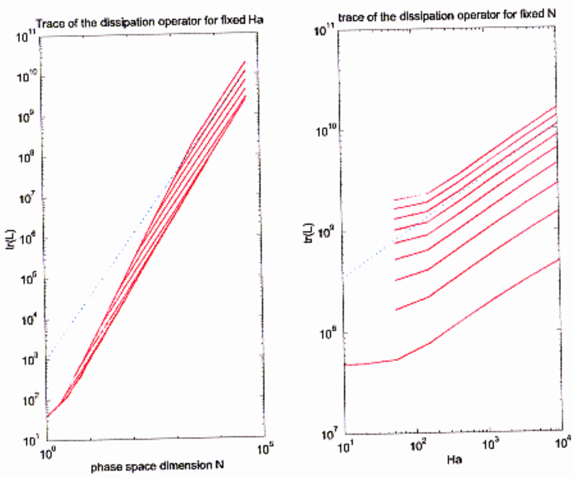

Figure 2: Solid line: numerical calculation. Dotted: Strong field approximation. The mismatch of the slopes is due to two dimensional state for low N (left), and to quasi-2d state for low Ha (right)

The shape of the iso-λ_{max} is determined by the ratio $\frac{N}{Ha^2}$. Intuitively, it indicates the relative importance of forcing versus dissipation (as a higher inertia tends to generate more modes, and hence, increase the dimension of the attractor). We notice that the smaller this number, the more modes are concentrated outside of the a cone of axis (Oz). This behaviour has been pointed out both experimentally ([1] and theoretically [4] for real flows, for which a strong magnetic field is known to confine turbulent modes outside the Joule cone. For dominating electro-magnetic effects, the Joule cone extends to the whole space except from the horizontal plane (kx, ky) : the flow becomes two-dimensional. This also occurs in the eigenvalue problem where two-dimensional modes appear to be the less dissipative ones. This allows us to find out whether the set of N eigenmode is purely two-dimensional.
In the case of a Joule cone-shaped ($\frac{N}{Ha^2} < 1$) distribution of a high number of modes ($N >> 1$) . An analytical expression for the trace of the dissipation can be found, by replacing the sum over the N eigenvalues by an integral.

$$\mathrm{Tr}(\mathbf{D}_N) \leq -cHa^{\frac{1}{2}}N^{\frac{3}{2}} \tag{7}$$

where c is any real constant of little relevance for our purpose. The associated maximum wavenumbers in the z-direction and in the orthogonal direction are given by:

$$k_{x_{max}} = k_{y_{max}} = cN^{\frac{1}{4}}Ha^{\frac{1}{4}} \tag{8a}$$

$$k_{z_{max}} = cN^{-\frac{1}{2}}Ha^{-\frac{1}{2}} \tag{8b}$$

and the flow is two dimensional if and only if $N < cHa$. The properties of the eigenmodes of the dissipation operator and those of the real flow exhibit some striking similarities. We shall exhibit more of them using the full result on the estimate for the attractor dimension.

4 bounds on turbulent MHD flow quantities

We shall now derive an estimate for the attractor dimension of the Navier-Stokes equation on a periodical domain. To this end, we add (4) and (7) in order to get an upper bound for the expansion rate of the Volume of any N-dimensional subset located in the vicinity of the attractor. We recall that the attractor dimension is the smallest value of the integer N for which this expansion rate is negative. The results are plotted on figure 3 in the general case. In the case $N >> 1$ and $\frac{N}{Ha^2} < 1$, equations (8a-8b) yield an analytical upper bound for the attractor dimension, as well as upper bounds for the

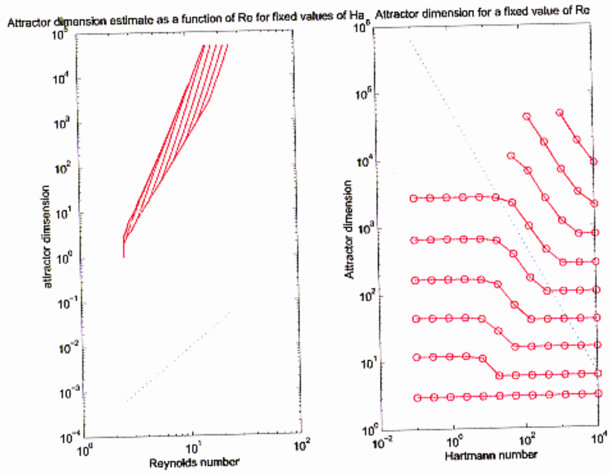

Figure 3: attractor dimension as a function of Re (left) and Ha (right). dotted: strong field approximation. solid: numerics.

maximum wavenumbers:

$$d_{att} \leq cRe^4 Ha^{-1} \tag{9a}$$

$$k_{x_{max}} \leq cRe \tag{9b}$$

$$k_{z_{max}} \leq cRe^2 Ha^{-1} \tag{9c}$$

To estimate the sharpness of these bounds, we shall compare them to the heuristic results obtained under the usual assumptions of 3d MHD turbulence. Indeed, it is 2 assumed that a balance is established at each scale k between inertia and Lorentz force. This leads to a power density spectrum varying as k^{-3}. Now assuming that small scales result from the balance between viscosity and inertia yields an order of magnitude for the latter:

$$k_{x_{max}} \sim cRe^{\frac{1}{2}} \tag{10a}$$

$$k_{z_{max}} \sim cReHa^{-1} \tag{10b}$$

The estimates appear to be loose when powers of the Reynolds number are compared. This is because it is difficult to have a sharp bound for the 3d inertial terms. However, the exponent of the Hartmann number is the same in both estimates and heuristic approximations, which stems from the fact that our estimate for the dissipation is derived from an achieved bound, and therefore, is the sharpest possible. This suggests that the modes minimising the dissipation are actually relevant to the real flow. This latter idea is also supported by the variations of d_{att} with Ha plotted on figure 3 in the general case. Indeed, the latter can be split in three zones. For low values of Ha, the attractor dimension hardly depends on Ha. This corresponds to a quasi-isotropic flow where electro-magnetic effects are negligible and for which the dissipation is viscous. For higher values of Ha, the attractor dimension decreases at a rate corresponding to (9a). In this region, Joule effect is the main dissipation mechanism, and is provoked by electric current loops tending to damp velocity differences between planes orthogonal to the magnetic field. For higher values of Ha, d_{att} becomes again independent of Ha. This happens when the damping mechanism involved at more moderate values of Ha dominates all other effects so that the flow is two-dimensional. Joule dissipation then disappears and in the absence of walls surrounding the fluid domain, the only remaining dissipation mechanism is viscous friction between column-like vortices, of rotation axis aligned with the magnetic field.

5 conclusion

Though they are not solution of the motion equations, the eigenmodes of the dissipation operator exhibit some strong similarities with what is known from the real flow. As these properties are derived under the only assumption that the solutions of the Navier-Stoles equations are regular, this gives some strong support to the assumptions on which former heuristic results rely. However, the estimates obtained might be improved if the estimate for the inertial terms is improved. Also, more physical behaviour such as boundary layer velocity profiles could be recovered by performing some similar study with classical wall boundary conditions on the planes orthogonal to the magnetic field.

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