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DNS of MHD turbulence in liquid metals based on the least dissipative modes

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We investigate numerically the incompressible Navier-Stokes equations under an externally imposed magnetic field. The results obtained for the simplified geometry of a three dimensional periodic box and show a lower computational complexity and stronger relationship between the physical reality and its numerical representation.

1 Problem formulation

We consider the case of a space periodic flow in a $3D$ cubic box Ω of size L under imposed homogeneous and steady magnetic field $\mathbf{B} = B_0 \cdot \mathbf{e}_z$. In the frame of the low-Rm approximation, the governing equations can be reduced to a single one involving the velocity $\mathbf{u}$ and pressure p only (see [3] and [4]). Using a reference velocity U and characteristic length L_{ref} we shall write it in a non dimensional form as

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \frac{1}{\text{Re}}(\nabla^2\mathbf{u} - \text{Ha}^2\nabla^{-2}\frac{\partial^2\mathbf{u}}{\partial z^2}) + \mathbf{f}(\mathbf{x},t), \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{1}$$

where following notations are used $\text{Ha} = L_{\text{ref}} B_0 \sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number and $\text{Re} = \frac{UL_{\text{ref}}}{\nu}$ is the Reynolds number, $\mathbf{u}(\mathbf{x},t) = (u(\mathbf{x},t), v(\mathbf{x},t), w(\mathbf{x},t))$ is the velocity-vector of the flow, $\mathbf{f}(\mathbf{x},t)$ is the external forcing, $\mathbf{x} = (x,y,x)$ is the spatial variable, t is the time, ρ is the density, p is the pressure, ν is the viscosity, σ is the electrical conductivity, B_0 is the imposed magnetic field. Periodic boundary conditions and zero initial condition $\mathbf{u}(\mathbf{x},0) = 0$ completely determine the problem.

We present numerical simulation results using pseudo-spectral method based on a decomposition of velocity $\mathbf{u}$ over the orthonormal basis of eigenfunctions $\mathbf{v_k}$ of the linear operator $D_{Ha} = \nabla^2 - \text{Ha}^2\nabla^{-2}\frac{\partial^2}{\partial z^2}$ of the problem (1). These eigenfunctions are a subset of the Fourier modes space used in the standard DNS schemes (see [5]). The aim is to show that properly chosen subset of least dissipative modes reduces the costs of the numerical simulations without loosing precision. It makes sense to consider eigenvalues $\lambda_{\mathbf{k}}$ which represents the rate of dissipation of mode $\mathbf{k}$

$$\lambda_{\mathbf{k}} = \lambda_{(k_x,k_y,k_z)} = -(k_x^2 + k_y^2 + k_z^2) - Ha^2\frac{k_z^2}{k_x^2 + k_y^2 + k_z^2}. \tag{2}$$

Since $\lambda_{\mathbf{k}} < 0$, $\lambda_{\mathbf{k}}$ can be arranged by growing dissipation so the spectral decomposition of $\mathbf{u}$ can be written as $\mathbf{u} = \sum_{\lambda_{\mathbf{k}} < \lambda^{\max}} c_{\lambda_{\mathbf{k}}}\mathbf{v}_{\lambda_{\mathbf{k}}}$, where $\lambda^{\max}$ defines the maximum resolution required to resolve the flow completely. This yields a natural spectral parameter $\lambda_{\mathbf{k}}$ that already incorporates anisotropy. In the case of $Ha = 0$, $|\lambda_{\mathbf{k}}|^{1/2}$ reduces to $||\mathbf{k}||$ which is the usual spectral parameter in non-MHD isotropic turbulence. As mentioned by [2], the set of least dissipative eigenmodes of D_{Ha} required to describe the flow exhibit the rate of anisotropy expected for such flow previous heuristic consideration. In short, one could see $\lambda_{\mathbf{k}}$ as an anisotropic generalization of the usual $\mathbf{k}$-sequence.

2 Numerical results

For the simulations, we set the following constant values $L = 0.1$ m, $\nu = 3.4\cdot 10^{-7}$ m^2/s, $\rho = 6.4\cdot 10^3$ kg/m^3, $\sigma = 3.46\cdot 10^6$ Ω^{-1}m^{-1}, $B_0 = 0.02$ T and flow is forced in the following way. A constant forcing is applied to Fourier modes with wave numbers $\mathbf{k_f} = (6,6,0), (7,7,0), (9,9,0)$

$$\mathbf{f}(\mathbf{x},t) = f_0\frac{L_{ref}}{U^2}\sum_{\mathbf{k_f}}\left(\sin(k_{f_x}\frac{2\pi}{L}x)\cos(k_{f_y}\frac{2\pi}{L}y)\mathbf{e}_x + \cos(k_{f_x}\frac{2\pi}{L}x)\sin(k_{f_y}\frac{2\pi}{L}y)\mathbf{e}_y\right), \tag{3}$$

that tends to generate a 2D flow. Since the numerical algorithm would not otherwise allow the solution of the problem to be 3D, we add a small constant excitation εf_0 in the circles $||\mathbf{k} - \mathbf{k_f}|| < 2$, with $\varepsilon \sim 10^{-3}$. Several simulations were done with

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different numerical resolutions $n_x \times n_y \times n_z$ and force amplitudes f_0. Figure 1 shows the time-averaged energy distributions of the 3D anisotropic MHD flow in the $(k_\perp, k_z)$-plane (here $k_\perp = \sqrt{k_x^2 + k_y^2}$) for fixed $f_0 = 0.0022$. The first picture corresponds to the $Re = 153$ and highest resolution $128 \times 128 \times 128$ which is necessary to resolve the flow according to the Kolmogorov length scale $k_{\max} = Re^{3/4} \approx 45$. The second picture is done with the same 128^3 resolution but all the modes $\mathbf{v_k}$ which correspond to the $|\lambda_{\mathbf{k}}| > 68$ are set to be zero. And the last one shows the energy of the flow resolved with 64^3 and cutoff for $\lambda_{\mathbf{k}} \leq 32$. Thus, the last run uses as much as 8 times less modes as the first classical run. The first

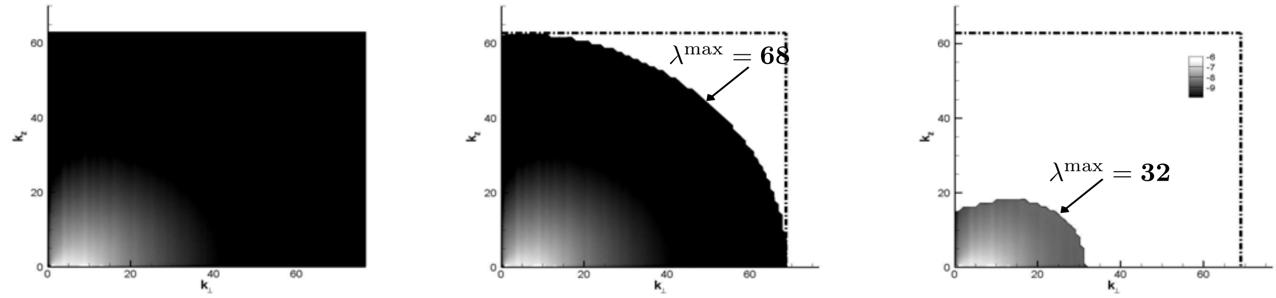

Fig. 1 Logarithmical energy distribution in the $(k_\perp, k_z)$-plane. Darker dots correspond to low energy modes. *Left* : Traditional spectral resolution involves 128^3 modes. *Centre* : resolution is 128^3 and cut-off for $|\lambda| \leq 68$ and *Right* : resolution is 64^3 and cut-off for $|\lambda| \leq 32$.

picture on Figure 2 shows the energy λ-spectrum $E(\lambda)$. One can see that spectra obtained in all three runs are very close (the same can be observed on the dissipation spectra). This shows that a cut off of the type $|\lambda_{\mathbf{k}}| < |\lambda^{\max}|$ achieves a good enough precision, even though the Kolmogorov scale do not resolved. Indeed, [2] suggested that the latter should be replaced by $k_\perp^{\max} = C_1 Re^{1/2}$ and $k_z^{\max} = C_2 \frac{Re}{Ha}$. We arbitrarily choose $\lambda^{\max}$ so that the energy contained in the $|\lambda_{\mathbf{k}}| < |\lambda^{\max}|$ modes represents 97% of the total energy. Next, in order to define $\lambda^{\max}$ we have done simulation for different values of f_0. The picture in the centre of Figure 2 shows the summation of dissipation energy in the dependency of λ. The black points in the curve are associated with $\lambda^{\max}$ and mean the 97%-level of the total energy. As can be seen in Figure 2 on the right, the corresponding values of $\lambda^{\max}$ are growing as $\sim G^{1/3}$, where G is the Grashof number. It appears that even for the low values of Re considered here, C_1 and C_2 remain almost constants and equal to 10 and 5 respectively, which confirms and quantifies the scaling from [2] and [1]. For further parametric study more points should be considered, in particular at higher Re and at

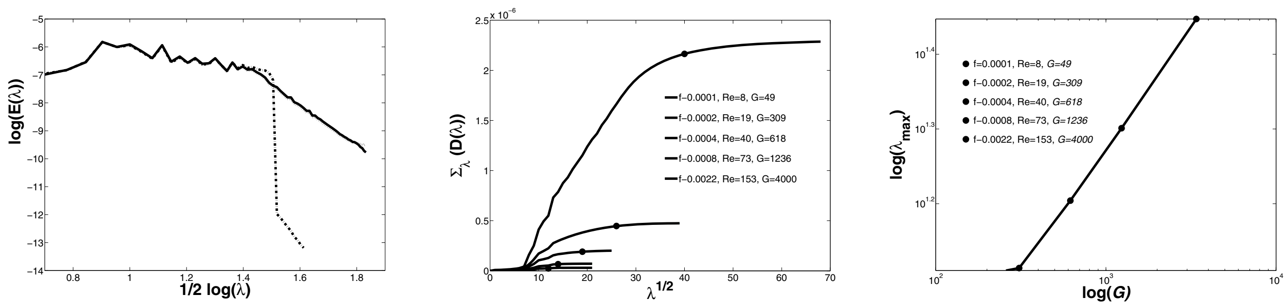

Fig. 2 *Left:* the energy spectra, solid line corresponds to 128^3 resolution, dashed line is 128^3 and cut off for $\lambda^{\max} = 68$; dash-doted line is 64^3 and $\lambda^{\max} = 68$. *Centre:* Summation of the dissipation energy. *Right* the Grashof dependence for $\lambda^{\max}$.

different magnetic fields B_0 in order to test scaling laws. But it can be already concluded now, that Kolmogorov scaling laws are very often pessimistic and much higher then necessary. The above presented λ-cutoff and corresponding λ-estimate can be considered as more realistic estimate of the number of modes required to resolve MHD turbulence completely.

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