Spectral study of the magnetohydrodynamic turbulence under imposed magnetic field

V. Dymkou and A. Pothérat

Applied Mathematics Research Centre,
Faculty of Engineering and Computing,
Coventry University, Coventry, UK
vitali.dymkou@coventry.ac.uk
alban.potherat@coventry.ac.uk

Abstract: We investigate numerically the non-linear equations of quasi-static magnetohydrodynamics (MHD) forced turbulence. The presented numerical approach is new compared to the existing classical approaches to turbulence. The results obtained for the simplified geometry of a three dimensional periodic box and show a lower computational complexity and stronger relationship between the physical reality and its numerical representation. This allows us to obtain new information on the flow's structure and size of the smallest scales.

1 Introduction

All type of spectral methods assume the existence of a series of orthogonal functions, in the sense that the function u, which defines the model, can be expanded in to the orthogonal functions series of the form

$$u = \sum_i c_i u_i, \qquad (1)$$

where u_i form a basis of the corresponding functional space and c_i are time dependent coefficients. It means that depending on the problem, a different type of decomposition can be used for Direct Numerical Simulation (DNS). Fourier series and Chebyshev polynomials are widely used for this purpose. The main advantage of these bases lies in their simple numerical implementation, for instance for integrating or differentiating. Also, it allows us to construct a general algorithm to solve a wide variety of problems in the same way. However, these sets of functions don't take into account the physical properties of the system under consideration directly. This may result into an excessive number of functions needed to determine a solution. As a consequence, availability of high-speed processors and computing power is the crucial factor for numerical implementation. Another possible way is to obtain the basis directly from the problem.

2 Problem formulation

We consider the case of a spatially periodic, incompressible, conducting fluid in a $3D$ cubic box Ω of size $L = L_{box}$ under imposed homogeneous and steady magnetic field B_0 aligned with the vertical direction e_z. In the frame of the low-Rm

approximation, so the governing equations can be reduced to a single one involving the velocity and pressure only (see [3] and [4]). Using a reference velocity U and characteristic length L_{ref} we shall write it in a non dimensional form as

$$\begin{aligned} &\frac{\partial}{\partial t}u(x,t)+(u\cdot\nabla)u+\nabla p=\nabla^2 u-Ha^2\nabla^{-2}\frac{\partial^2 u}{\partial z^2}+Gf(x,t), \\ &\nabla\cdot u=0, \end{aligned} \qquad (2)$$

where following notations are used $\mathrm{H}a=L_{ref}B_0\sqrt{\frac{\sigma}{\rho\nu}}$ is the Hartmann number and $G=\frac{L_{ref}^{3-d/2}}{\nu^2}\|f\|$ is the Grashof number (d is number of spatial dimensions). $\mathrm{R}e=\frac{UL_{ref}}{\nu}$ is the Reynolds number, $u(x,t)$ is the velocity-vector of the flow, $f(x,t)$ is the external forcing, $x=(x,y,x)$ is the spatial variable, t is time, ρ is the density, p is the pressure, ν is the viscosity, σ is the electrical conductivity, B_0 is the imposed magnetic field. Additionally, we will use another non-dimensional parameter Reynolds number $Re=\frac{UL_{int}}{\nu}$ based on integral length scale L_{int} (see [5]). The addition of periodic boundary conditions and zero initial condition $u(x,0)=0$ completely determine the problem.

We present numerical study using pseudo-spectral method based on a decomposition of the velocity u over the orthonormal basis of the eigenfunctions v_k of the linear operator $D_{Ha}=\nabla^2-\mathrm{H}a^2\nabla^{-2}\frac{\partial^2}{\partial z^2}$, which corresponds to the linear part of the problem (2). These eigenfunctions are in a subset of the Fourier space used in the standard DNS schemes (see [5]). The aim is to show that properly chosen subset of least dissipative modes reduces the costs of the numerical simulations without loosing precision. It makes sense to consider eigenvalues λ_k which represents the rate of dissipation of mode k

$$\lambda_k=\lambda_{(k_x,k_y,k_z)}=-(k_x^2+k_y^2+k_z^2)-Ha^2\frac{k_z^2}{k_x^2+k_y^2+k_z^2}. \qquad (3)$$

Since $\lambda_k<0$, λ_k can be arranged by growing dissipation so the spectral decomposition of u can be written as $u=\sum_{\lambda_k<\lambda^{max}}c_{\lambda_k}v_{\lambda_k}$, where λ^{max} defines the maximum resolution required to resolve the flow completely. This yields a natural spectral parameter λ_k that already incorporates anisotropy. In the case of $Ha=0$, $|\lambda_k|^{1/2}$ reduces to $\|k\|$ which is the usual spectral parameter in non-MHD isotropic turbulence. As mentioned by [2], the set of least dissipative eigenmodes of D_{Ha} required to describe the flow exhibits the rate of anisotropy expected for such flow from previous heuristic consideration. In short, one could see λ_k as an anisotropic generalization of the usual k-sequence.

3 Numerical results

For the simulations, we set the following constant values $L = 0.1$ m, $\nu = 3.4 \cdot 10^{-7}$ m^2/s, $\rho = 6.4 \cdot 10^3$ kg/m^3, $\sigma = 3.46 \cdot 10^6$ Ω^{-1}m^{-1}, $B_0 = 0.02$ T and the flow is forced in the following way: a constant forcing is applied to Fourier modes with wave numbers $k_f = (6,6,0),(7,7,0)$ and $(9,9,0)$

$$f(x,t) = f_0 \frac{L_{ref}^2}{\nu^2} \sum_{k_f} (\sin(k_{f_x} \frac{2\pi}{L} x) \cos(k_{f_y} \frac{2\pi}{L} y) e_x + \cos(k_{f_x} \frac{2\pi}{L} x) \sin(k_{f_y} \frac{2\pi}{L} y) e_y), \quad (4)$$

where f_0 is the force amplitude. Such a choice of the force tends to generate a 2D flow. Since the numerical algorithm would not otherwise allow the solution of the problem to be 3D at all, we add a small constant excitation εf_0 inside the disks $\| k - k_f \| < 2$, with $\varepsilon :\ 10^{-3}$. There are several reasons for this choice: first, the forcing has to be a combination of the set of modes used for the expansion. A z-independent forcing can be used to simulate both 2D and 3D flows. The second reason is that this type of constant 2D forcing strongly resembles that obtained in liquid metal experiments by injecting electric current though metallic electrodes embedded in insulating walls.

Several simulations were done with different numerical resolutions $n_x \times n_y \times n_z$ and Grashof numbers G. Figure 1 shows the time-averaged energy distributions of the 3D anisotropic MHD flow in the $(k_\perp, k_z)$-plane (here $k_\perp = \sqrt{k_x^2 + k_y^2}$) for fixed $G = 27193$ ($Re = 92$). The first picture corresponds to the $Re = 92$ and highest resolution $128 \times 128 \times 128$ which is necessary to resolve the flow according to the Kolmogorov length scale $k_{\max} = 1.5 Re^{3/4} \approx 45$. The second picture is done with the same 128^3 resolution but all the modes v_k which correspond to the $|\lambda_k| > 68$ are set to be zero. And the last one shows the energy of the flow resolved with 64^3 and cutoff for $\lambda_k \leq 32$. Thus, the last run uses as much as 8 times less modes as the first classical run.

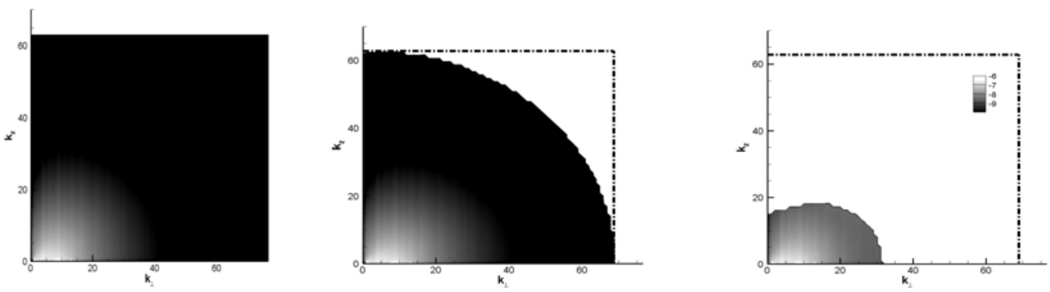

Figure 1: Logarithmical energy distribution in the $(k_\perp, k_z)$-plane. Darker dots correspond to low energy modes. *Left* : Traditional spectral resolution involves 128^3 modes. *Centre* : resolution is 128^3 and all the modes v_k which correspond to the $|\lambda_k|^{1/2} > 68$ are set to zero (cut-off for $|\lambda|^{1/2} \leq 68$) and *Right* : resolution is 64^3 and cut-off for $|\lambda|^{1/2} \leq 32$.

The first picture on Figure 2 shows the energy λ-spectrum $E(\lambda)$. One can see that spectra obtained in all three runs are very close (the same can be observed on the dissipation spectra). This shows that a cut off of the type $|\lambda_k|<|\lambda^{max}|$ achieves a good enough precision, even though the Kolmogorov scale is not resolved. Indeed, [2] suggested that the latter should be replaced by $k_{\perp}^{max}=C_1 Re^{1/2}$ and $k_z^{max}=C_2\dfrac{Re}{Ha}$. Then, in order to define λ^{max} we have done simulation for different Grashof numbers G (or Re). The picture in the centre of Figure 2 shows the the summation of the total energy $E(\lambda)$ in the dependency of λ. The presence of a plateau on the right hand side of the graph confirms that the flows are over-resolved, as it indicates that modes corresponding to these values of λ carry almost no energy nor produce any dissipation. We arbitrarily choose λ^{max} so that the energy contained in the $|\lambda_k|<|\lambda^{max}|$ modes represents 97%, 90% and 80% of the total energy, respectively. As can be seen in Figure 2 on the right, in all cases the corresponding values of λ^{max} are growing as : $G^{1/4}$. It appears that even for the low values of Re considered here, C_1 and C_2 remain almost constants and equal to 1 and 3/2 respectively, whichconfirms and quantifies the scaling from [2] and [3].

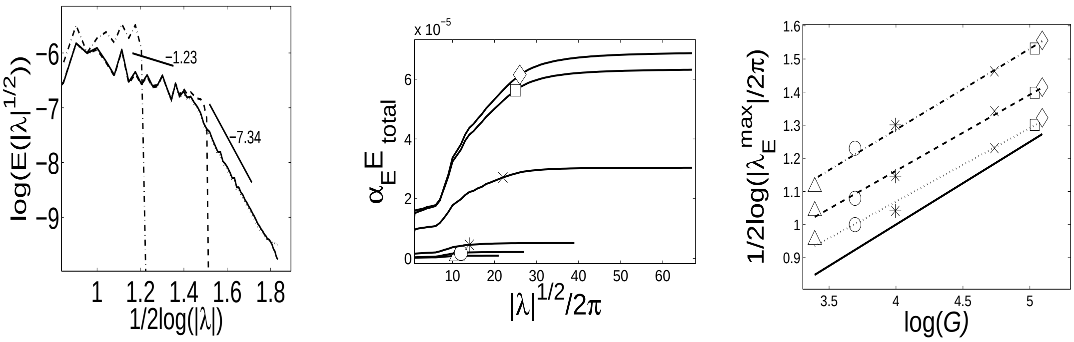

Figure 2: *Left:* the energy spectra in λ-shells, solid line corresponds to 128^3 resolution, dashed line is 128^3 and cut off for $\lambda^{max}=68$; dash-doted line is 64^3 and $\lambda^{max}=68$. *Centre:* Summation of the energy for the different values of G. Symbols on the curves are associated with the value of λ^{max} and mean the 97% of the total energy. *Right* Grashof dependencies for $|\lambda^{max}|^{1/2}$. Dotted, dashed and dashed-dotted lines present the different values of $\alpha_E(\lambda)$, namely 80%, 90% and 97% of the total energy respectively.

4 Conclusion

For further parametric study more points should be considered, in particular at higher Re and at different magnetic fields B_0 in order to refine the scaling laws for λ^{max}. But it can be already concluded now, that Kolmogorov scaling laws are very often pessimistic and much higher then necessary. The above presented λ-cutoff and corresponding λ-estimate can be considered as more realistic estimate for the number of modes required to resolve MHD turbulence completely.

References

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